



Matrices

INTRODUCTION

Matrices owe their origin to various linear problems, the most important of which consists in the nature of solutions of systems of simultaneous linear equations. In 1857, Arthur Cayley developed the properties of matrices as a pure algebraic structure. These days, matrices are used in a large number of disciplines such as Algebra, Geometry, Statistics, Physics, Chemistry, Psychology, Economics and Industrial management etc. In the present chapter, we shall learn about matrices and shall confine ourselves to the study of basic laws of matrix algebra.

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MATRICES

Consider the systems of simultaneous linear equations:

$$\left. \begin{array}{l} 2x - 3y = 7 \\ 5x + 4y = 11 \end{array} \right\}$$

It is a system of two linear equations in two unknowns x, y ; and ...(i)

$$\left. \begin{array}{l} 2x - 7y + 3z = 8 \\ 3x + 0y + 6z = -11 \end{array} \right\} \quad \text{...(ii)}$$

is a system of two linear equations in three unknowns x, y, z . In the first case, to know L.H.S. we need only know the array $\begin{bmatrix} 2 & -3 \\ 5 & 4 \end{bmatrix}$ consisting of the coefficients of x and y , in this particular order.

Similarly, the array $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$ gives all information about R.H.S. Thus, the system of equations in (i) is

completely specified by the two arrays of numbers $\begin{bmatrix} 2 & -3 \\ 5 & 4 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$. Similarly, the system of

equations in (ii) is fully specified by the two arrays $\begin{bmatrix} 2 & -7 & 3 \\ 3 & 0 & 6 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ -11 \end{bmatrix}$.

Each array of the above type is known as a *matrix*. The numbers, of which each matrix is constituted, are called the *elements* or *entries*. In the above examples, all elements are real numbers. In fact, the elements can be complex numbers or polynomials.

For each element of a matrix, its value as well as its position are both important. It means that a matrix is an *ordered array* of elements.

In the (i) system of equations, the array $[2 \ -3]$ specifies the left side of the first equation and the array $[5 \ 4]$ that of the second equation. Each such horizontal array of a matrix is called a *row* of the matrix.

Similarly, the array $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ specifies the coefficients of x and the array $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ specifies the coefficients of y in the (i) system of equations. Each such vertical line is called a *column* of the matrix.

Thus, the matrix $\begin{bmatrix} 2 & -3 \\ 5 & 4 \end{bmatrix}$ consists of two rows and two columns or briefly is a 2×2 matrix, while the matrix $\begin{bmatrix} 2 & -7 & 3 \\ 3 & 0 & 6 \end{bmatrix}$ corresponding to the (ii) system of equations is a matrix with two rows and three columns or briefly a 2×3 matrix. The matrices corresponding to the right side of the above systems (i) and (ii) are 2×1 matrices.

Matrix. A rectangular arrangement of mn elements in the form of an ordered set of m rows, each row consisting of an ordered set of n elements is called an $m \times n$ matrix ($m \times n$ is read as m by n).

Such an arrangement is enclosed in the bracket [] or parenthesis (). Each of the mn elements constituting the matrix is called an **element** or **entry** of the matrix. To locate the position of a particular element of a matrix, we have to specify the number of row and that of the column in which the element occurs.

An element appearing in the i th row and the j th column is referred as (i, j) th element or (i, j) th entry. Usually, a matrix is denoted by a capital letter and an element of a matrix by a small letter of the *alphabet* along with two suffixes, the first one indicating the number of row and the latter one that of the column in which the element appears. Thus, an $m \times n$ matrix may be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}.$$

In compact form, this matrix is written as

$$A = [a_{ij}]_{m \times n}, 1 \leq i \leq m, 1 \leq j \leq n.$$

A is called a matrix of order (or type) $m \times n$.

For example : $A = \begin{bmatrix} 2 & -3 & 1 & \sqrt{2} \\ 0 & 1 & -2 & 7 \\ 3 & 4 & 9 & -\sqrt{3} \end{bmatrix}$ is a matrix of order 3×4 .

It has 3×4 i.e. 12 elements. It has 3 rows and 4 columns. Here $a_{11} = 2$, $a_{23} = -2$, $a_{31} = 3$, $a_{34} = -\sqrt{3}$ etc.

If $a_{ij} \in \mathbf{R}$ (set of reals), we say that A is a matrix over $\mathbf{R}$ or A is a real matrix. Similarly, if $a_{ij} \in \mathbf{C}$ (complex numbers), then A is a *complex* matrix. In this book, all matrices will have real entries.

The array $[a_{i1} \ a_{i2} \ a_{i3} \ \dots \ a_{in}]$ is the i th row and the array $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \dots \\ a_{mj} \end{bmatrix}$ is the j th column of the matrix A .

Comparable matrices

Two matrices are said to be **comparable** iff each one of them contains as many rows and columns as the other.

Equal matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be **equal** (written as $A = B$) iff

- (i) they are of the same order i.e. number of rows in A = number of rows in B and number of columns in A = number of columns in B , and
- (ii) their corresponding elements are equal i.e. $a_{ij} = b_{ij}$ for all i and j .

For example : the matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ are equal iff $a = \alpha, b = \beta, c = \gamma$ and $d = \delta$.

Types of Matrices

1. Row matrix

A matrix having only one row is called a **row matrix**. Thus, $A = [1 \ 5 \ 3 \ -7]$ is a 1×4 row matrix.

2. Column matrix

A matrix having only one column is called a **column matrix**. Thus, $A = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}$ is a 3×1 column matrix.

3. Zero matrix

An $m \times n$ matrix whose each element is zero is called a **zero matrix** or a **null matrix** of order $m \times n$. Thus, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of order 2×3 .

4. Square matrix

A matrix in which the number of rows equals the number of columns, say n , is called a **square matrix** of order n or an **n -rowed square matrix**.

The elements a_{ij} of a square matrix $A = [a_{ij}]$ for which $i = j$ are called the **diagonal elements** and the line along which these elements lie is called the **principal diagonal** or simply the **diagonal** of the matrix. Thus, the elements $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are the diagonal elements of the square matrix $A = [a_{ij}]_{n \times n}$.

For example : $\begin{bmatrix} 2 & 3 & -1 \\ 4 & 7 & 9 \\ 0 & 1 & 3 \end{bmatrix}$ is a square matrix of order 3 and its diagonal elements are 2, 7, 3.

5. Diagonal matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is called a **diagonal matrix** iff all its non-diagonal elements are zero. The diagonal elements may or may not be zero.

For example : $[4]$, $\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are diagonal matrices of order 1, 2 and 3 respectively.

A diagonal matrix of order n is written as diagonal $[a_{11}, a_{22}, a_{33}, \dots, a_{nn}]$.

6. Scalar matrix

A diagonal matrix in which all diagonal elements are equal is called a **scalar matrix**. Thus in a scalar matrix, $A = [a_{ij}]_{n \times n}$, we have

$$a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ k, & \text{when } i = j. \end{cases}$$

For example : $[3]$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ are scalar matrices of order 1, 2 and 3 respectively.

7. Identity matrix

A scalar matrix in which each diagonal element is unity is called an **identity matrix** or a **unit matrix**. Thus in an identity matrix, $A = [a_{ij}]_{n \times n}$, we have

$$a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ 1, & \text{when } i = j. \end{cases}$$

For example : $[1]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1, 2 and 3 respectively.

An identity matrix of order n is usually denoted by I_n .

8. Triangular matrix

(i) **Upper Triangular matrix.** A square matrix $A = [a_{ij}]_{n \times n}$ is called an **upper triangular matrix** iff $a_{ij} = 0$ for all $i > j$ i.e. iff all elements below the principal diagonal are zero.

(ii) **Lower Triangular matrix.** A square matrix $A = [a_{ij}]_{n \times n}$ is called a **lower triangular matrix** iff $a_{ij} = 0$ for all $i < j$ i.e. iff all elements above the principal diagonal are zero.

(iii) **Triangular matrix.** A matrix, upper or lower triangular is called a **triangular matrix**.

A triangular matrix $A = [a_{ij}]_{n \times n}$ is called **strictly triangular** iff $a_{ii} = 0$ for all $i = 1, 2, 3, \dots, n$.

For example : $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 7 & 0 & 0 \\ 4 & 0 & 0 \\ -1 & 2 & 9 \end{bmatrix}$ are upper triangular and lower triangular matrices of order 3 respectively.

ILLUSTRATIVE EXAMPLES

Example 1. A is a matrix of the type 3×5 and R is a row of A , then what is the type of R as a matrix?

Solution. Since A is a matrix of type 3×5 , therefore, each row of A contains 5 elements. Hence, R is a row matrix of the type 1×5 .

Example 2. If a matrix has 12 elements, what are the possible orders it can have? What, if it has 7 elements?

Solution. We know that if a matrix is of order $m \times n$, then it has mn elements. Thus to find all possible orders of a matrix with 12 elements, we shall find all ordered pairs (of natural numbers) whose product is 12.

All possible ordered pairs are

$$(1, 12), (12, 1), (2, 6), (6, 2), (3, 4), (4, 3).$$

Hence, the possible orders are

$$1 \times 12, 12 \times 1, 2 \times 6, 6 \times 2, 3 \times 4 \text{ and } 4 \times 3.$$

Further, if the matrix has 7 elements, then the possible orders are 1×7 and 7×1 .

Example 3. Let $A = \begin{bmatrix} 3 & 7 & -1 \\ 0 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 9 & 0 \\ 3 & -5 & 6 \end{bmatrix}$. Find the sum of (2, 2) entries of A and B .

Solution. Here (2, 2) entry of A is 2 and (2, 2) entry of B is -5 .

$\therefore$ The sum of (2, 2) entries of A and $B = 2 + (-5) = -3$.

Example 4. Construct a 2×3 matrix whose elements a_{ij} are given by

(i) $a_{ij} = 2i - 3j$

(ii) $a_{ij} = i \cdot j$

Solution. (i) $a_{ij} = 2i - 3j$, $i = 1, 2$ and $j = 1, 2, 3$

$$\therefore a_{11} = 2 \cdot 1 - 3 \cdot 1 = -1, \quad a_{12} = 2 \cdot 1 - 3 \cdot 2 = -4, \quad a_{13} = 2 \cdot 1 - 3 \cdot 3 = -7;$$

$$a_{21} = 2 \cdot 2 - 3 \cdot 1 = 1, \quad a_{22} = 2 \cdot 2 - 3 \cdot 2 = -2, \quad a_{23} = 2 \cdot 2 - 3 \cdot 3 = -5.$$

Hence, the required matrix is $\begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \end{bmatrix}$.

(ii) $a_{ij} = i \cdot j$, $i = 1, 2$ and $j = 1, 2, 3$

$$\therefore a_{11} = 1 \cdot 1 = 1, \quad a_{12} = 1 \cdot 2 = 2, \quad a_{13} = 1 \cdot 3 = 3;$$

$$a_{21} = 2 \cdot 1 = 2, \quad a_{22} = 2 \cdot 2 = 4, \quad a_{23} = 2 \cdot 3 = 6.$$

Hence, the required matrix is $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$.

Example 5. Construct a 3×3 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = \frac{(i+2j)}{5}$.

Solution. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$a_{ij} = \frac{(i+2j)}{5}; i = 1, 2, 3; j = 1, 2, 3$$

$$\therefore a_{11} = \frac{1+2 \times 1}{5} = \frac{3}{5}, \quad a_{12} = \frac{1+2 \times 2}{5} = 1, \quad a_{13} = \frac{1+2 \times 3}{5} = \frac{7}{5},$$

$$a_{21} = \frac{2+2 \times 1}{5} = \frac{4}{5}, \quad a_{22} = \frac{2+2 \times 2}{5} = \frac{6}{5}, \quad a_{23} = \frac{2+2 \times 3}{5} = \frac{8}{5},$$

$$a_{31} = \frac{3+2 \times 1}{5} = 1, \quad a_{32} = \frac{3+2 \times 2}{5} = \frac{7}{5}, \quad a_{33} = \frac{3+2 \times 3}{5} = \frac{9}{5}.$$

Hence, $A = \begin{bmatrix} \frac{3}{5} & 1 & \frac{7}{5} \\ \frac{4}{5} & \frac{6}{5} & \frac{8}{5} \\ 1 & \frac{7}{5} & \frac{9}{5} \end{bmatrix}$.

Example 6. Find the value of $x - y + z$ if $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$.

Solution. By definition of equality of matrices, we get

$$x + y + z = 9 \quad \dots(i)$$

$$x + z = 5 \quad \dots(ii)$$

$$y + z = 7 \quad \dots(iii)$$

Subtracting (ii) from (i), we get $y = 4$.

From (iii), we get $4 + z = 7 \Rightarrow z = 3$.

From (ii), we get $x + 3 = 5 \Rightarrow x = 2$.

$$\therefore x - y + z = 2 - 4 + 3 = 1.$$

Example 7. Find the values of a, b, c and d from the matrix equation :

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

Solution. Given $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

By definition of equality of matrices, we get

$$a-b=-1, 2a+c=5, 2a-b=0, 3c+d=13.$$

$$\text{Now } a-b=-1, 2a-b=0 \Rightarrow a-2a=-1, b=2a$$

$$\Rightarrow a=1, b=2 \times 1=2.$$

$$2a+c=5 \Rightarrow 2 \times 1+c=5 \Rightarrow c=3.$$

$$3c+d=13 \Rightarrow 3 \times 3+d=13 \Rightarrow d=4.$$

$$\text{Hence, } a=1, b=2, c=3, d=4.$$

Example 8. Can you find the values of x and y so that the matrices

$$\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix} \text{ and } \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix} \text{ may be equal?}$$

Solution. $\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix} = \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$

$$\Rightarrow 3x+7=0, 5=y-2, y+1=8 \text{ and } 2-3x=4$$

$$\Rightarrow x=-\frac{7}{3}, y=7, y=7 \text{ and } x=-\frac{2}{3}.$$

We note that $x=-\frac{7}{3}$ and $x=-\frac{2}{3}$ cannot be true simultaneously, therefore, the given matrices cannot be equal.

Example 9. Find x, y, a and b if

$$\begin{bmatrix} 3x+4y & 6 & x-2y \\ a+b & 2a-b & -3 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 5 & -5 & -3 \end{bmatrix}.$$

Solution. By definition of equality of matrices, we get

$$3x+4y=2 \quad \dots(i)$$

$$x-2y=4 \quad \dots(ii)$$

$$a+b=5 \quad \dots(iii)$$

$$2a-b=-5 \quad \dots(iv)$$

To find x, y ;

Multiply (ii) by 2 and then add it to (i), we get $5x=10 \Rightarrow x=2$.

$$\therefore \text{ From (i), } 3 \times 2 + 4y = 2 \Rightarrow 4y = -4 \Rightarrow y = -1.$$

To find a, b ;

Adding (iii) and (iv), we get $3a=0 \Rightarrow a=0$.

$$\therefore \text{ From (iii), } 0+b=5 \Rightarrow b=5.$$

Hence, $x=2, y=-1, a=0, b=5$.

Example 10. What is the number of all possible matrices of order 3×3 with each entry 0 or 1?

Solution. Since a matrix of order 3×3 has 9 elements i.e. 9 entries and each entry can be filled in 2 ways (by 0 or 1), therefore, the total number of ways of filling these entries is 2^9 i.e. 512.

Corresponding to every different entry, we get a different matrix.

∴ The total number of all possible matrices of order 3×3 with each entry 0 or 1 is 512.

EXERCISE 3.1

1. Identify the type of matrices given below and write the order of each matrix:

$$(i) A = [2 \ 3] \quad (ii) B = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$(iii) C = \begin{bmatrix} 2 & 5 \\ -1 & 2 \\ 4 & 0 \end{bmatrix}$$

$$(iv) D = [1 \ 3 \ -4]$$

$$(v) E = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$$

$$(vi) P = \begin{bmatrix} 0 & -4 & 3 \\ 1 & 0 & -7 \\ 2 & 2 & 0 \end{bmatrix}$$

$$(vii) Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(viii) R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(ix) X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(x) Y = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$(xi) Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

2. (i) If $A = \begin{bmatrix} -9 & 4 & -3 \\ -1 & 0 & 4 \\ 2 & 2 & 0 \end{bmatrix}$, find $a_{22} + a_{32}$.

(ii) If $B = \begin{bmatrix} -5 & 7 & 2 \\ 8 & 0 & 7 \\ -2 & 1 & 3 \end{bmatrix}$, find $b_{21} + b_{32} - b_{13}$.

3. If A is a matrix of type $p \times q$ and R is a row of A, then what is the type of R as a matrix?
4. If A is a column matrix with 5 rows, then what type of matrix is a row of A?
5. If a matrix has 18 elements, what the possible orders it can have? What, if it has 11 elements?
6. If A is a 3×3 matrix whose elements are given by $a_{ij} = \frac{1}{3} | -3i + j |$, write the value of a_{23} .
7. Construct a 2×2 matrix $A = [a_{ij}]$ whose elements a_{ij} are given by:
- (i) $a_{ij} = 2i - j$ (ii) $a_{ij} = |2i - 3j|$.
8. Construct a 2×3 matrix $B = [b_{ij}]$ whose elements b_{ij} are given by:
- (i) $b_{ij} = i - 3j$ (ii) $b_{ij} = (i + 2j)^2$.

9. (i) If $\begin{bmatrix} x + 2y & -y \\ 3x & 4 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 6 & 4 \end{bmatrix}$, find the values of x and y .

(ii) Find the value of x , if $\begin{bmatrix} 3x + y & -y \\ 2y - x & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -5 & 3 \end{bmatrix}$.

(iii) If $\begin{bmatrix} x - y & z \\ 2x - y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, find the value of $(x + y)$.

(iv) If $\begin{bmatrix} x + y & x + 2 \\ 2x - y & 16 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 1 & 3y + 1 \end{bmatrix}$, then write the value of $(y - x)$.

10. Find the values of a, b, c and d from the following equations :

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

11. Find x, y, a and b if $\begin{bmatrix} 2x-3y & a-b & 3 \\ 1 & x+4y & 3a+4b \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 6 & 29 \end{bmatrix}$.

12. (i) Write the number of all possible matrices of order 2×3 with each entry 1 or 2.
(ii) Write the number of all possible matrices of order 2×2 with each entry 1, 2 or 3.

Answers

1. (i) Row matrix, 1×2 (ii) Column matrix, 3×1
(iii) Rectangular matrix, 3×2 (iv) Row matrix, 1×3
(v) Square matrix, 2×2 (vi) Square matrix, 3×3
(vii) Null matrix, 2×2 (viii) Null matrix, 2×3
(ix) Unit matrix, 3×3 (x) Scalar matrix, 3×3
(xi) Diagonal matrix, 3×3
2. (i) 2 (ii) 7
3. $1 \times q$ 4. 1×1 5. $1 \times 18, 18 \times 1, 2 \times 9, 9 \times 2, 3 \times 6, 6 \times 3, 1 \times 11, 11 \times 1$
6. 1 7. (i) $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}$ 8. (i) $\begin{bmatrix} -2 & -5 & -8 \\ -1 & -4 & -7 \end{bmatrix}$ (ii) $\begin{bmatrix} 9 & 25 & 49 \\ 16 & 36 & 64 \end{bmatrix}$
9. (i) $x=2, y=-3$ (ii) 1 (iii) 3 (iv) 2
10. $a=1, b=2, c=3, d=4$ 11. $x=2, y=1, a=3, b=5$ 12. (i) 64 (ii) 81

OPERATIONS ON MATRICES

Addition of Matrices

If A and B are two matrices of the same order, we say that these are **compatible** (or **conformable**) for addition and their sum $A+B$ is the matrix obtained by adding the corresponding elements of A and B .

Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then $A+B = [a_{ij} + b_{ij}]_{m \times n}$, $1 \leq i \leq m, 1 \leq j \leq n$.

For example : if $A = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 0 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 9 & -11 \\ 3 & 1 & -2 \end{bmatrix}$,

$$\text{then } A+B = \begin{bmatrix} 2+4 & 3+9 & -1+(-11) \\ 5+3 & 0+1 & 7+(-2) \end{bmatrix} = \begin{bmatrix} 6 & 12 & -12 \\ 8 & 1 & 5 \end{bmatrix}.$$

Note

If A and B are not of the same order, then $A+B$ is not defined, and we say that A and B are not compatible for the sum $A+B$.

Subtraction of Matrices

If A and B are two matrices of the same order, we say that these are **compatible for subtraction** and their difference $A-B$ is the matrix obtained by subtracting the elements of B from the corresponding elements of A .

Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then $A-B = [a_{ij} - b_{ij}]_{m \times n}$, $1 \leq i \leq m, 1 \leq j \leq n$.

For example : if $A = \begin{bmatrix} 4 & -7 & -2 \\ 5 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 & -5 \\ -9 & 6 & -3 \end{bmatrix}$

$$\text{then } A - B = \begin{bmatrix} 4-2 & -7-4 & -2-(-5) \\ 5-(-9) & 3-6 & 0-(-3) \end{bmatrix} = \begin{bmatrix} 2 & -11 & 3 \\ 14 & -3 & 3 \end{bmatrix}.$$

Negative of a Matrix

Let $A = [a_{ij}]_{m \times n}$ be a matrix, then $-A = [-a_{ij}]_{m \times n}$ is called the **negative** of matrix A .

$$\text{For example, if } A = \begin{bmatrix} 2 & -3 & 0 \\ 5 & 7 & -6 \end{bmatrix}, \text{ then } -A = \begin{bmatrix} -2 & 3 & 0 \\ -5 & -7 & 6 \end{bmatrix}.$$

Properties of Matrix Addition

1. Addition of matrices is commutative

i.e. if A and B are matrices of the same order, then $A + B = B + A$.

2. Addition of matrices is associative

i.e. if A , B and C are matrices of the same order, then $(A + B) + C = A + (B + C)$.

3. Existence of additive identity

For any matrix A , there exists a null matrix O of the same order such that $A + O = A = O + A$.

4. Existence of additive inverse

For any matrix $A = [a_{ij}]_{m \times n}$, there exists a matrix $-A = [-a_{ij}]_{m \times n}$ such that $A + (-A) = O = (-A) + A$.

Multiplication of a Matrix by a Scalar (number)

If k is any scalar (number) and A is any matrix, then the matrix obtained by multiplying each element of A by k is called the **scalar multiple** of A by k and is denoted by kA .

Thus, if $A = [a_{ij}]_{m \times n}$, then $kA = [ka_{ij}]_{m \times n}$.

$$\text{For example : if } A = \begin{bmatrix} 2 & 5 & 0 \\ -3 & 7 & 9 \end{bmatrix},$$

$$\text{then } 3A = \begin{bmatrix} 3 \times 2 & 3 \times 5 & 3 \times 0 \\ 3 \times (-3) & 3 \times 7 & 3 \times 9 \end{bmatrix} = \begin{bmatrix} 6 & 15 & 0 \\ -9 & 21 & 27 \end{bmatrix}.$$

Properties of Scalar Multiplication

If A , B are two matrices of the same order and k, l are any scalars (numbers), then

- (i) $(k + l)A = kA + lA$ (ii) $k(A + B) = kA + kB$ (iii) $(kl)A = k(lA) = l(kA)$
 (iv) $1A = A$ (v) $(-1)A = -A$.

ILLUSTRATIVE EXAMPLES

Example 1. Let $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}$.

Verify that $A + (B + C) = (A + B) + C$.

$$\text{Solution. } A + (B + C) = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} + \left(\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \right) = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 1 & -1 \end{bmatrix}$$

$$\text{and } (A + B) + C = \left(\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \right) + \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 1 & -1 \end{bmatrix}.$$

$$\therefore A + (B + C) = (A + B) + C.$$

Example 2. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 & 5 \\ 2 & 3 & -4 \end{bmatrix}$, find $2A - 3B$.

$$\begin{aligned} \text{Solution. } 2A - 3B &= 2 \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} - 3 \begin{bmatrix} 0 & -1 & 5 \\ 2 & 3 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 & -15 \\ -6 & -9 & 12 \end{bmatrix} \\ &= \begin{bmatrix} 2+0 & 4+3 & 6-15 \\ 4-6 & 6-9 & 2+12 \end{bmatrix} = \begin{bmatrix} 2 & 7 & -9 \\ -2 & -3 & 14 \end{bmatrix}. \end{aligned}$$

Example 3. If $A = \text{diagonal } [1, -2, 5]$ and $B = \text{diagonal } [3, 0, -4]$, then find $3A - 2B$.

Solution. Given $A = \text{diagonal } [1, -2, 5]$ and $B = \text{diagonal } [3, 0, -4]$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}.$$

$$\begin{aligned} \therefore 3A - 2B &= 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix} - 2 \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 15 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{bmatrix} \\ &= \begin{bmatrix} 3-6 & 0-0 & 0-0 \\ 0-0 & -6-0 & 0-0 \\ 0-0 & 0-0 & 15-(-8) \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 23 \end{bmatrix} \\ &= \text{diagonal } [-3, -6, 23]. \end{aligned}$$

Example 4. If $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} + 2 \begin{bmatrix} 2x \\ 3y \end{bmatrix} = 3 \begin{bmatrix} 7 \\ -3 \end{bmatrix}$, find the values of x and y .

$$\text{Solution. Given } \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} + 2 \begin{bmatrix} 2x \\ 3y \end{bmatrix} = 3 \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} + \begin{bmatrix} 4x \\ 6y \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix} \Rightarrow \begin{bmatrix} x^2 + 4x \\ y^2 + 6y \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$

By definition of equality of matrices, we get

$$x^2 + 4x = 21 \text{ and } y^2 + 6y = -9$$

$$\Rightarrow x^2 + 4x - 21 = 0 \text{ and } y^2 + 6y + 9 = 0$$

$$\Rightarrow (x-3)(x+7) = 0 \text{ and } (y+3)^2 = 0$$

$$\Rightarrow x = 3, -7 \text{ and } y = -3.$$

Hence, $x = 3, y = -3$ or $x = -7, y = -3$.

Example 5. Given $3 \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$, find the values of x, y, z and w .

$$\text{Solution. Given } 3 \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}$$

By definition of equality of matrices, we get

$$3x = x + 4 \quad \dots(i)$$

$$3y = 6 + x + y \quad \dots(ii)$$

$$3z = -1 + z + w \quad \dots(iii)$$

$$3w = 2w + 3 \quad \dots(iv)$$

From (i), we get $2x = 4 \Rightarrow x = 2$.

From (ii), we get $2y = 6 + 2 \Rightarrow 2y = 8 \Rightarrow y = 4$.

From (iv), we get $w = 3$

From (iii), we get $2z = -1 + 3 \Rightarrow 2z = 2 \Rightarrow z = 1$.

Hence, $x = 2, y = 4, z = 1, w = 3$.

Example 6. If A, B and C are three matrices of the same order, then

$$B = C \Rightarrow A + B = A + C.$$

Solution. Since A, B and C are of the same order, $A + B$ and $A + C$ are also of the same order.

Now $B = C \Rightarrow$ corresponding elements of B and C are equal

$\Rightarrow (i, j)$ th element of $B = (i, j)$ th element of C

$$\Rightarrow b_{ij} = c_{ij} \Rightarrow a_{ij} + b_{ij} = a_{ij} + c_{ij} \quad (\because b = c \Rightarrow a + b = a + c)$$

$\Rightarrow (i, j)$ th element of $(A + B) = (i, j)$ th element of $(A + C)$

$\Rightarrow$ corresponding elements of $(A + B)$ and $(A + C)$ are equal

$$\Rightarrow A + B = A + C.$$

Remark

In view of the above result, if A, B and C are three matrices of the same order, then $A - B = C \Rightarrow A = B + C$.

Example 7. If A, B and C are three matrices such that $A + B = A + C$, then $B = C$.

Solution. Since $A + B = A + C$

$\Rightarrow A, B$ and C are of the same order

$\Rightarrow -A, A, B$ and C are of the same order.

Now $A + B = A + C$ (given)

$$\Rightarrow (-A) + (A + B) = (-A) + (A + C) \quad (\text{using example 6})$$

$$\Rightarrow (-A + A) + B = (-A + A) + C \quad (\text{Associative law})$$

$$\Rightarrow O + B = O + C \Rightarrow B = C.$$

Example 8. Find a matrix A such that $2A - 3B + 5C = O$, where $B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$.

Solution. Given $2A - 3B + 5C = O$

$$\Rightarrow 2A = 3B - 5C + O \Rightarrow 2A = 3B - 5C$$

$$\Rightarrow 2A = 3 \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix} - 5 \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} -6 & 6 & 0 \\ 9 & 3 & 12 \end{bmatrix} - \begin{bmatrix} 10 & 0 & -10 \\ 35 & 5 & 30 \end{bmatrix}$$

$$\Rightarrow 2A = \begin{bmatrix} -6-10 & 6-0 & 0-(-10) \\ 9-35 & 3-5 & 12-30 \end{bmatrix} \Rightarrow A = \frac{1}{2} \begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix} = \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}.$$

Example 9. If X and Y are 2×2 matrices, then solve the following matrix equations for X and Y :

$$2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}, 3X + 2Y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix}.$$

Solution. Given $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \quad \dots(i)$

$$\text{and} \quad 3X + 2Y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix} \quad \dots(ii)$$

Multiplying (i) by 2 and (ii) by 3, we get

$$4X + 6Y = 2 \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 0 \end{bmatrix} \quad \dots(iii)$$

$$9X + 6Y = 3 \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 3 & -15 \end{bmatrix} \quad \dots(iv)$$

Subtracting (iii) from (iv), we get

$$5X = \begin{bmatrix} -6 & 6 \\ 3 & -15 \end{bmatrix} - \begin{bmatrix} 4 & 6 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} -10 & 0 \\ -5 & -15 \end{bmatrix}$$

$$\Rightarrow X = \frac{1}{5} \begin{bmatrix} -10 & 0 \\ -5 & -15 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix}.$$

$$\text{From (ii), we get} \quad 2Y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix} - 3X$$

$$\Rightarrow 2Y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix} - 3 \begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix} - \begin{bmatrix} -6 & 0 \\ -3 & -9 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 4 & 4 \end{bmatrix}$$

$$\Rightarrow Y = \frac{1}{2} \begin{bmatrix} 4 & 2 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}.$$

Example 10. Two booksellers A and B sell the books of Mathematics and Applied Mathematics. In the month of March, bookseller A sold 250 books of Mathematics and 400 books of Applied Mathematics whereas bookseller B sold 230 books of Mathematics and 425 books of Applied Mathematics. In the month of April, bookseller A sold 550 books of Mathematics and 300 books of Applied Mathematics and bookseller B sold 270 books of Mathematics and 450 books of Applied Mathematics.

Represent the given information into matrix form and find the total sale for both the booksellers in the month of March and April using matrix Algebra.

Solution. Books sold by both the booksellers in the month of March can be represented in the matrix form as given below:

$$P = \begin{matrix} & \begin{matrix} \text{Maths} & \text{Applied} \\ & \text{Maths} \end{matrix} \\ \begin{pmatrix} 250 & 400 \\ 230 & 425 \end{pmatrix} & \begin{matrix} \text{Bookseller A} \\ \text{Bookseller B} \end{matrix} \end{matrix}$$

Also, books sold by both the booksellers in the month of April can be represented in the matrix form as given below:

$$Q = \begin{matrix} & \begin{matrix} \text{Maths} & \text{Applied} \\ & \text{Maths} \end{matrix} \\ \begin{pmatrix} 550 & 300 \\ 270 & 450 \end{pmatrix} & \begin{matrix} \text{Bookseller A} \\ \text{Bookseller B} \end{matrix} \end{matrix}$$

$$\therefore \text{Total sale} = P + Q = \begin{pmatrix} 250 & 400 \\ 230 & 425 \end{pmatrix} + \begin{pmatrix} 550 & 300 \\ 270 & 450 \end{pmatrix}$$

$$\Rightarrow P + Q = \begin{pmatrix} \text{Maths} & \text{Applied Maths} \\ 800 & 700 \\ 500 & 875 \end{pmatrix} \begin{matrix} \text{Bookseller A} \\ \text{Bookseller B} \end{matrix}$$

Hence, total sale by bookseller A in the months of March and April are 800 books of Maths and 700 books of Applied Maths, whereas by bookseller B are 500 books of Maths and 875 books of Applied Maths.

EXERCISE 3.2

1. If $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$, find the following :

(i) $A + B$

(ii) $3A - C$

2. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then find $2A - B$.

3. If $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$, then find the matrix A.

4. If $A = \text{diagonal } [1, -2, 5]$, $B = \text{diagonal } [3, 0, -4]$ and $C = \text{diagonal } [-2, 7, 0]$, then find $A + 2B - 3C$.

5. If $A = \begin{bmatrix} 2 & -3 \\ 5 & 0 \end{bmatrix}$ and $kA = \begin{bmatrix} 8 & 3a \\ -2b & c \end{bmatrix}$, find the values of k, a, b and c .

6. If $x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, find the values of x and y .

7. If $3 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 6 \\ -1 & 2d \end{bmatrix} + \begin{bmatrix} 4 & a+b \\ c+d & 3 \end{bmatrix}$, find a, b, c and d .

8. Find X if $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$.

9. Find matrices X and Y if

$$2X - Y = \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} \text{ and } X + 2Y = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 1 & -7 \end{bmatrix}.$$

10. Find a matrix X such that $3A - 2B + X = O$, where

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}, B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}.$$

Answers

1. (i) $\begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$

2. $\begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}$

4. diagonal $[13, -23, -3]$

5. $k = 4, a = -4, b = -10, c = 0$

6. $x = 3, y = -4$

7. $a = 2, b = 4, c = 1, d = 3$ 8. $\begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$

9. $X = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & -1 \end{bmatrix}, Y = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

10. $\begin{bmatrix} -16 & -4 \\ 3 & -5 \end{bmatrix}$

MULTIPLICATION OF MATRICES

Let A and B be two matrices, then the product AB of A and B is defined if and only if the number of columns in A is equal to the number of rows in B , and we say that these are **compatible** (or **conformable**) for the product AB . If A is a matrix of order $m \times n$ and B is a matrix of order $n \times p$, then AB is a matrix of order $m \times p$ and is defined as $AB = [c_{ik}]_{m \times p}$ where

$$(i, k)\text{th element of } AB = \begin{cases} \text{sum of the products of the elements of the } i\text{th row of } A \\ \text{with the corresponding elements of the } k\text{th column of } B \end{cases}$$

i.e. if $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{n \times p}$, then the product AB is defined as the matrix $C = [c_{ik}]_{m \times p}$ where

$$c_{ik} = \sum_{j=1}^n a_{ij}b_{jk}, 1 \leq i \leq m, 1 \leq k \leq p.$$

In the product AB , the matrix A is called the **pre-factor** and the matrix B is called the **post-factor**. Also we say that the matrix A has been **post-multiplied** by the matrix B and the matrix B has been **pre-multiplied** by the matrix A .

For example :

(i) Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 5 & -2 \end{bmatrix}$.

Since A is of order 2×2 and B is of order 2×2 , therefore, AB is defined and it is a matrix of order 2×2 .

$$AB = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 1.0 + (-1).5 & 1.1 + (-1).(-2) \\ 2.0 + 3.5 & 2.1 + 3.(-2) \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 15 & -4 \end{bmatrix}.$$

Also B is of order 2×2 and A is of order 2×2 , therefore, BA is defined and it is a matrix of order 2×2 .

$$BA = \begin{bmatrix} 0 & 1 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0.1 + 1.2 & 0.(-1) + 1.3 \\ 5.1 + (-2).2 & 5.(-1) + (-2).3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -11 \end{bmatrix}.$$

Note that $AB \neq BA$.

(ii) Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -3 & 5 \end{bmatrix}$.

Since A is of order 2×3 and B is of order 3×2 , therefore, AB is defined and it is a matrix of order 2×2 .

$$\begin{aligned} AB &= \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 2.1 + (-1).2 + 3.(-3) & 2.0 + (-1).1 + 3.5 \\ 0.1 + 1.2 + 4.(-3) & 0.0 + 1.1 + 4.5 \end{bmatrix} = \begin{bmatrix} -9 & 14 \\ -10 & 21 \end{bmatrix}. \end{aligned}$$

Also B is of order 3×2 and A is of order 2×3 , therefore, BA is defined and it is a matrix of order 3×3 .

$$\begin{aligned} BA &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1.2 + 0.0 & 1.(-1) + 0.1 & 1.3 + 0.4 \\ 2.2 + 1.0 & 2.(-1) + 1.1 & 2.3 + 1.4 \\ -3.2 + 5.0 & -3.(-1) + 5.1 & -3.3 + 5.4 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -1 & 10 \\ -6 & 8 & 11 \end{bmatrix}. \end{aligned}$$

Note that $AB \neq BA$.

(iii) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$.

Since A is of order 3×2 and B is of order 2×2 , therefore, AB is defined and it is a matrix of order 3×2 .

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1.1 + 1.2 & 1.2 + 1.2 \\ 0.1 + 2.2 & 0.2 + 2.2 \\ 1.1 + 1.2 & 1.2 + 1.2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 4 \\ 3 & 4 \end{bmatrix}.$$

Again B is of order 2×2 and A is of order 3×2 , therefore, BA is not defined.

Remark

From the above examples, we observe that the *multiplication of matrices* is not commutative. This does not mean that $AB \neq BA$ for every pair of matrices A, B for which AB and BA are defined.

For example, if $A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ then $AB = \begin{bmatrix} 6 & 0 \\ 0 & -5 \end{bmatrix} = BA$.

In fact, multiplication of diagonal matrices of the same order is always commutative.

Properties of Matrix Multiplication

If A, B, C are any three (suitable) matrices and λ is any scalar, then

1. Multiplication is distributive over addition

i.e. if A, B, C are suitable matrices, then

(i) $A(B + C) = AB + AC$ and (ii) $(A + B)C = AC + BC$.

The above laws are called *distributive laws*.

2. Multiplication of matrices is associative

i.e. if A, B, C are suitable matrices, then $(AB)C = A(BC)$.

3. If A, B are suitable matrices and λ is a scalar, then $\lambda(AB) = (\lambda A)B = A(\lambda B)$.

4. Existence of multiplicative identity

If $A = [a_{ij}]$ is an $m \times n$ matrix, then $I_m A = A = A I_n$.

5. The product of any matrix and a suitable null matrix is a null matrix

If $A = [a_{ij}]$ is an $m \times n$ matrix, then $O_{p \times m} A = O_{p \times n}$ and $A O_{n \times q} = O_{m \times q}$.

Powers of a Square Matrix

Let A be a square matrix of order n , then AA makes sense and it is also a square matrix of order n . We define:

$$A^1 = A$$

$$A^2 = AA$$

$$\dots \dots \dots$$

$$\dots \dots \dots$$

$$A^m = A^{m-1}A = AA^{m-1}, \text{ for all positive integers } m.$$

Matrix Polynomial

Let $f(x) = a_0x^m + a_1x^{m-1} + \dots + a_{m-1}x + a_m$ be a polynomial in x and A be a square matrix of order n , then

$f(A) = a_0A^m + a_1A^{m-1} + \dots + a_{m-1}A + a_mI_n$ is called a matrix polynomial in A . Thus, to obtain $f(A)$, replace x by A in $f(x)$ and the constant term is multiplied by the identity matrix of the order equal to that of A .

The polynomial equation $f(x) = 0$ is said to be satisfied by the matrix A iff $f(A) = O$, where O is a null matrix of order $n \times n$.

For example : if $f(x) = 2x^2 - 3x + 7$ and A is a square matrix of order 3, then $f(A) = 2A^2 - 3A + 7I_3$.

ILLUSTRATIVE EXAMPLES

Example 1. If $A = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}$ and $BA = (b_{ij})$, find $b_{21} + b_{32}$.

Solution. Given $A = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}$ and $BA = (b_{ij})$

$$\Rightarrow \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix} = (b_{ij}).$$

By definition of multiplication of matrices, we have

$$b_{21} = 4.1 + 5.(-4) = 4 - 20 = -16 \text{ and}$$

$$b_{32} = 2.(-2) + 1.2 = -4 + 2 = -2.$$

$$\therefore b_{21} + b_{32} = (-16) + (-2) = -18.$$

Example 2. If $A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$, verify that $A(B + C) = AB + AC$.

$$\begin{aligned} \text{Solution. } A(B + C) &= A \left(\begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} \right) = A \begin{bmatrix} 3+4 & 1+7 \\ -1+2 & 0+1 \\ 4+1 & 2+(-1) \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 1 & 1 \\ 5 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2.7 + 0.1 + (-3).5 & 2.8 + 0.1 + (-3).1 \\ 1.7 + 4.1 + 5.5 & 1.8 + 4.1 + 5.1 \end{bmatrix} = \begin{bmatrix} -1 & 13 \\ 36 & 17 \end{bmatrix} \dots(i) \end{aligned}$$

$$AB = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2.3 + 0.(-1) + (-3).4 & 2.1 + 0.0 + (-3).2 \\ 1.3 + 4.(-1) + 5.4 & 1.1 + 4.0 + 5.2 \end{bmatrix} = \begin{bmatrix} -6 & -4 \\ 19 & 11 \end{bmatrix}$$

$$\text{and } AC = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4 + 0.2 + (-3).1 & 2.7 + 0.1 + (-3).(-1) \\ 1.4 + 4.2 + 5.1 & 1.7 + 4.1 + 5.(-1) \end{bmatrix} = \begin{bmatrix} 5 & 17 \\ 17 & 6 \end{bmatrix}$$

$$\therefore AB + AC = \begin{bmatrix} -6 & -4 \\ 19 & 11 \end{bmatrix} + \begin{bmatrix} 5 & 17 \\ 17 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -6+5 & -4+17 \\ 19+17 & 11+6 \end{bmatrix} = \begin{bmatrix} -1 & 13 \\ 36 & 17 \end{bmatrix} \quad \dots(ii)$$

From (i) and (ii), we get

$$A(B + C) = AB + AC.$$

Example 3. Find the values of a and b for which the following hold :

$$\begin{bmatrix} a & b \\ -a & 2b \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\text{Solution. Given } \begin{bmatrix} a & b \\ -a & 2b \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2.a + b.(-1) \\ (-a).2 + 2b.(-1) \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 2a - b \\ -2a - 2b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2a - b = 5 \\ -2a - 2b = 4 \text{ i.e. } a + b = -2 \end{cases} \quad \dots(i)$$

$$\dots(ii)$$

Adding (i) and (ii), we get $3a = 3 \Rightarrow a = 1$

From (ii), $1 + b = -2 \Rightarrow b = -3$

Hence, $a = 1, b = -3$.

Example 4. If matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ and $A^2 = pA$, then write the value of p .

$$\text{Solution. } A^2 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 2.2 + (-2)(-2) & 2.(-2) + (-2).2 \\ (-2).2 + 2.(-2) & (-2)(-2) + 2.2 \end{bmatrix} = \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}$$

$$\text{Given } A^2 = pA \Rightarrow \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} = p \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} = \begin{bmatrix} 2p & -2p \\ -2p & 2p \end{bmatrix}$$

$$\Rightarrow 8 = 2p, -8 = -2p \Rightarrow p = 4.$$

Note

However, if we get different values of p then the question is meaningless.

Example 5. If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, find the value(s) of α for which $A^2 = B$.

Solution. We have, $A^2 = B$

$$\Rightarrow \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha\alpha + 0.1 & \alpha.0 + 0.1 \\ 1.\alpha + 1.1 & 1.0 + 1.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha^2 & 0 \\ \alpha + 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$\Rightarrow \alpha^2 = 1 \text{ and } \alpha + 1 = 5$$

$$\Rightarrow \alpha = \pm 1 \text{ and } \alpha = 4, \text{ which is not possible.}$$

Hence, there is no value of α for which $A^2 = B$.

Example 6. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then show that $A^2 - 3I = 2A$.

Solution. $A^2 = AA = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1.1 + 2.2 & 1.2 + 2.1 \\ 2.1 + 1.2 & 2.2 + 1.1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

$$\begin{aligned} \therefore A^2 - 2A - 3I &= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - 2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 5-2-3 & 4-4-0 \\ 4-4-0 & 5-2-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O \end{aligned}$$

$$\Rightarrow A^2 - 3I = 2A.$$

Example 7. Find the value of k if $M = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $M^2 - kM - I_2 = O$.

Solution. $M^2 = MM = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1.1 + 2.2 & 1.2 + 2.3 \\ 2.1 + 3.2 & 2.2 + 3.3 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}$

Given $M^2 - kM - I_2 = O$

$$\Rightarrow \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix} - k \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix} - \begin{bmatrix} k & 2k \\ 2k & 3k \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5-k-1 & 8-2k-0 \\ 8-2k-0 & 13-3k-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4-k & 8-2k \\ 8-2k & 12-3k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow 4-k=0, 8-2k=0, 8-2k=0 \text{ and } 12-3k=0$$

$$\Rightarrow k=4, k=4, k=4 \text{ and } k=4.$$

Hence, the value of k is 4.

Example 8. If $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find the values of x and y such that $A^2 + xI_2 = yA$.

Solution. $A^2 = AA = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 3.3 + 1.7 & 3.1 + 1.5 \\ 7.3 + 5.7 & 7.1 + 5.5 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$.

Given $A^2 + xI_2 = yA$

$$\Rightarrow \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix} + x \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = y \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix} + \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 16+x & 8+0 \\ 56+0 & 32+x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix}$$

$$\Rightarrow \begin{cases} 16+x=3y & \dots(i) & 8+0=y & \dots(ii) \\ 56+0=7y & \dots(iii) & 32+x=5y & \dots(iv) \end{cases}$$

From (ii), we get $y = 8$.

From (i), we get $16+x = 3 \times 8 \Rightarrow 16+x = 24 \Rightarrow x = 8$.

Note that these values of x and y satisfy (iii) and (iv).

Hence, $x = 8, y = 8$.

Example 9. If A is a square matrix such that $A^2 = A$, then find the value of $7A - (I + A)^3$, where I is the identity matrix.

Solution. $(I + A)^2 = (I + A)(I + A) = I^2 + IA + AI + A^2$
 $= I + A + A + A$ ($\because A^2 = A$ given)
 $= I + 3A$.

$$(I + A)^3 = (I + A)^2 (I + A) = (I + 3A)(I + A)$$

$$= I^2 + IA + 3AI + 3A^2 = I + A + 3A + 3A = I + 7A.$$

$$\therefore 7A - (I + A)^3 = 7A - (I + 7A) = -I.$$

Example 10. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, find a and b .

Solution. Given $(A + B)^2 = A^2 + B^2$

$$\Rightarrow (A + B)(A + B) = A^2 + B^2$$

$$\Rightarrow (A + B)A + (A + B)B = A^2 + B^2$$

$$\Rightarrow A^2 + BA + AB + B^2 = A^2 + B^2$$

$$\Rightarrow BA + AB = O$$

$$\Rightarrow \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix} + \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a-b+2 & -a+1 \\ 2a-2 & -b+4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow 2a-b+2=0, -a+1=0, 2a-2=0, -b+4=0$$

$$\Rightarrow 2a-b+2=0, a=1, a=1, b=4.$$

Note that $a = 1, b = 4$ satisfies $2a - b + 2 = 0$.

Hence, $a = 1, b = 4$.

Example 11. Find x , if $[x \ -5 \ -1] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$.

Solution. Given $[x \ -5 \ -1] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$

$$\Rightarrow [x+0-2 \ 0-10+0 \ 2x-5-3] \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$$

$$\Rightarrow [x-2 \ -10 \ 2x-8] \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$$

$$\Rightarrow [(x-2)x + (-10) \times 4 + (2x-8) \times 1] = [0]$$

$$\Rightarrow x^2 - 2x - 40 + 2x - 8 = 0 \Rightarrow x^2 = 48$$

$$\Rightarrow x = \pm 4\sqrt{3}.$$

Example 12. Find the matrix X for which $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$.

Solution. Since $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ are both square matrices of order 2, therefore, X must be square matrix of order 2.

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the given matrix equation becomes

$$\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5a+4c & 5b+4d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 5a+4c = 1 & \dots(i) \\ a+c = 1 & \dots(iii) \end{cases} \quad \begin{cases} 5b+4d = -2 & \dots(ii) \\ b+d = 3 & \dots(iv) \end{cases}$$

Solving (i) and (iii) simultaneously, we get $a = -3, c = 4$.

Solving (ii) and (iv) simultaneously, we get $b = -14, d = 17$.

$$\text{Hence, } X = \begin{bmatrix} -3 & -14 \\ 4 & 17 \end{bmatrix}.$$

Example 13. Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$. Find a matrix D such that $CD - AB = O$.

Solution. Since A, B, C are all square matrices of order 2, therefore, D must be a square matrix of order 2 for $CD - AB$ to be defined.

$$\text{Let } D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

$$\text{Given } CD - AB = O \Rightarrow CD = AB$$

$$\Rightarrow \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{bmatrix} = \begin{bmatrix} 10-7 & 4-4 \\ 15+28 & 6+16 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix}$$

By the definition of equality of matrices, we get

$$2a + 5c = 3 \quad \dots(i) \qquad 2b + 5d = 0 \quad \dots(ii)$$

$$3a + 8c = 43 \quad \dots(iii) \qquad 3b + 8d = 22 \quad \dots(iv)$$

Solving (i) and (iii), we get $a = -191$, $c = 77$.

Solving (ii) and (iv), we get $b = -110$, $d = 44$.

$$\therefore D = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}.$$

Example 14. Given that $X_{2 \times n}$, $Y_{3 \times k}$, $Z_{2 \times p}$, $W_{n \times 3}$ and $P_{p \times k}$ are matrices of specified order. What are the conditions:

(i) for n , k and p so that $3PY + 2WY$ is defined?

(ii) for the order of the matrix $2X - 3Z$?

Solution. (i) For multiplication of P and Y ,

number of columns of P = number of rows of Y

$$\Rightarrow k = 3$$

and order of $PY = p \times k$ i.e. $p \times 3$.

Also, for multiplication of W and Y ,

number of columns of W = number of rows of Y

$$\Rightarrow 3 = 3$$

$\therefore WY$ exists and order of $WY = n \times k = n \times 3$.

(i) Then, for addition of $3PY$ and $2WY$,

order of $3PY$ = order of $2WY$

$$\Rightarrow p \times 3 = n \times 3 \Rightarrow p = n.$$

(ii) $2X - 3Z$ is defined if

order of X = order of Z

$$\Rightarrow 2 \times n = 2 \times p \Rightarrow n = p.$$

Example 15. Cost of a pen and a notebook are ₹12 and ₹27 respectively. On a given day shopkeeper P sells 5 pens and 7 notebooks, whereas shopkeeper Q sells 6 pens and 4 notebooks. Find the money received by both the booksellers using matrix algebra.

Solution. Number of pens and notebooks sold by both the shopkeepers are represented by sale matrix

$$S = \begin{matrix} & \begin{matrix} \text{Pens} & \text{Notebooks} \end{matrix} \\ \begin{pmatrix} 5 & 7 \\ 6 & 4 \end{pmatrix} & \begin{matrix} \text{Shopkeeper P} \\ \text{Shopkeeper Q} \end{matrix} \end{matrix}$$

Also, cost of a pen and a notebook can be represented by a cost matrix

$$C = \begin{pmatrix} ₹12 \\ ₹27 \end{pmatrix} \begin{matrix} \text{Pen} \\ \text{Notebook} \end{matrix}$$

Now, the amount received by both the shopkeeper can be obtained by multiplying S and C .

$$\begin{aligned}\text{So, amount received} &= SC = \begin{pmatrix} 5 & 7 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 12 \\ 27 \end{pmatrix} \\ &= \begin{pmatrix} 60 + 189 \\ 72 + 108 \end{pmatrix} = \begin{pmatrix} 249 \\ 180 \end{pmatrix}.\end{aligned}$$

Hence, shopkeeper P received ₹ 249 and shopkeeper Q received ₹ 180.

EXERCISE 3.3

1. If $A = \begin{bmatrix} -1 & 2 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix}$, write the orders of AB and BA .
2. If A is a matrix of order 3×3 and B is a matrix of order 2×3 , can you find AB ?
3. If $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 4 & 2 \end{bmatrix}$ and $AB = [c_{ij}]$, find $c_{23} + c_{31}$.
4. If $\begin{bmatrix} x & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = O$, find the value of x .
5. If $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -9 & x \end{bmatrix}$, write the value of x .
6. Write the value of $x + y + z$ if $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.
7. If $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ and $A^2 = I$, find the value of $\alpha^2 + \beta\gamma$.
8. (i) If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, find AB and BA .
(ii) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $AB = BA$, then show that B is a scalar matrix.
Hint. (ii) Show that $a = b$.
9. If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $A^2 = B$, then find the value(s) of α .
10. If A is a square matrix such that $A^2 = I$, then find the simplified value $(A - I)^3 + (A + I)^3 - 7A$.
11. Write the following as a single matrix: $\begin{bmatrix} 3 & 2 & 5 \\ 7 & -4 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & -1 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 7 & -8 \\ 5 & 9 \end{bmatrix}$.
12. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -1 & 7 \end{bmatrix}$, find $3A^2 - 2B + I$.

13. Find the values of x and y , given that $\begin{bmatrix} x & y \\ 3y & x \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.
14. If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, show that $A^2 - 5A - 14I = O$.
15. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, then find k so that $A^2 = kA - 2I$.
16. If $A = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$, find $f(A)$, where $f(x) = x^2 - 2x + 3$.
17. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then find λ, μ so that $A^2 = \lambda A + \mu I$.
18. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, verify that $A^2 - 4A - 5I = O$.
19. If A and B are square matrices of the same order such that $AB = BA$, prove that
- (i) $(A + B)(A - B) = A^2 - B^2$
 - (ii) $(A + B)^2 = A^2 + 2AB + B^2$
 - (iii) $(A - B)^2 = A^2 - 2AB + B^2$
 - (iv) $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$
 - (v) $(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$.
20. Find x , if $\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O$.
21. Find a 2×2 matrix B such that $B \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = 6I_2$.
22. Two badminton teams A and B are staying in the same hotel. Team A has 2 male and 3 female players accompanied by 1 coach. Team B comprises of 1 male, 2 female players and 2 coaches. The daily requirement (calories and protein) for each person is as given below:

	Calories	Protein
Male player	2500	65 g
Female player	1900	50 g
Coach	2000	54 g

Use matrix algebra to calculate the total diet requirement of calories and protein for each team.

Answers

1. Order of AB is 1×1 and order of BA is 3×3 2. No 3. 10
4. 2 5. 13 6. 0 7. 1 8. (i) $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = BA$
9. 1 10. A 11. $\begin{bmatrix} 18 & 37 \\ 1 & 9 \end{bmatrix}$ 12. $\begin{bmatrix} 4 & -20 \\ 38 & -10 \end{bmatrix}$ 13. $x = 1, y = 1$
15. 1 16. $\begin{bmatrix} 12 & -4 \\ -6 & 8 \end{bmatrix}$ 17. $\lambda = 4, \mu = -1$ 20. $-2, -14$ 21. $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$
22. Team A: 12700 calories and 334 g protein; Team B: 10300 calories and 273 g protein

TRANSPOSE OF A MATRIX

Let $A = [a_{ij}]$ be an $m \times n$ matrix then **transpose** of A , denoted by A' or A^t , is the matrix obtained from A by interchanging its rows and columns. Clearly, A' is an $n \times m$ matrix whose (j, i) th element = (i, j) th element of A

i.e. $A' = [b_{ji}]$ where $b_{ji} = a_{ij}$, $1 \leq i \leq m$, $1 \leq j \leq n$.

For example : if $A = \begin{bmatrix} 2 & 5 & -1 \\ 7 & 4 & 0 \end{bmatrix}$ then $A' = \begin{bmatrix} 2 & 7 \\ 5 & 4 \\ -1 & 0 \end{bmatrix}$.

Properties of Transpose

Let A, B be suitable matrices, then

- (i) $(A')' = A$
- (ii) $(\lambda A)' = \lambda A'$, where λ is any scalar
- (iii) $(-A)' = -A'$
- (iv) $(A + B)' = A' + B'$
- (v) $(A - B)' = A' - B'$
- (vi) $(AB)' = B'A'$ (Reversal law)

ILLUSTRATIVE EXAMPLES

Example 1. If matrix $A = [1 \ 2 \ 3]$, write AA' where A' is the transpose of matrix A .

Solution. As matrix A is 1×3 , so matrix A' is 3×1 , therefore, AA' is a 1×1 matrix.

$$\begin{aligned} AA' &= [1 \ 2 \ 3][1 \ 2 \ 3]' = [1 \ 2 \ 3] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= [1.1 + 2.2 + 3.3] = [14]. \end{aligned}$$

Example 2. If $A = \begin{bmatrix} -1 & 5 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$, verify that

- (i) $(2A)' = 2A'$
- (ii) $(A + B)' = A' + B'$
- (iii) $(AB)' = B'A'$.

Solution. (i) $2A = 2 \begin{bmatrix} -1 & 5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 10 \\ 6 & 4 \end{bmatrix}$

$$\Rightarrow (2A)' = \begin{bmatrix} -2 & 10 \\ 6 & 4 \end{bmatrix}' = \begin{bmatrix} -2 & 6 \\ 10 & 4 \end{bmatrix}$$

$$\text{Also } 2A' = 2 \begin{bmatrix} -1 & 5 \\ 3 & 2 \end{bmatrix}' = 2 \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 10 & 4 \end{bmatrix}$$

From (1) and (2), we find that $(2A)' = 2A'$.

$$(ii) \ A + B = \begin{bmatrix} -1 & 5 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} -1+3 & 5+(-2) \\ 3+5 & 2+4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 8 & 6 \end{bmatrix}$$

$$\Rightarrow (A + B)' = \begin{bmatrix} 2 & 3 \\ 8 & 6 \end{bmatrix}' = \begin{bmatrix} 2 & 8 \\ 3 & 6 \end{bmatrix}$$

$$\begin{aligned} \text{Also } A' + B' &= \begin{bmatrix} -1 & 5 \\ 3 & 2 \end{bmatrix}' + \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}' = \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -1+3 & 3+5 \\ 5+(-2) & 2+4 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 3 & 6 \end{bmatrix} \end{aligned}$$

From (1) and (2), we find that $(A + B)' = A' + B'$.

$$(iii) AB = \begin{bmatrix} -1 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} -1.3 + 5.5 & (-1).(-2) + 5.4 \\ 3.3 + 2.5 & 3.(-2) + 2.4 \end{bmatrix} = \begin{bmatrix} 22 & 22 \\ 19 & 2 \end{bmatrix}$$

$$\Rightarrow (AB)' = \begin{bmatrix} 22 & 22 \\ 19 & 2 \end{bmatrix}' = \begin{bmatrix} 22 & 19 \\ 22 & 2 \end{bmatrix} \quad \dots(1)$$

$$\begin{aligned} \text{Also } B'A' &= \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}' \begin{bmatrix} -1 & 5 \\ 3 & 2 \end{bmatrix}' = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3.(-1) + 5.5 & 3.3 + 5.2 \\ (-2).(-1) + 4.5 & (-2).3 + 4.2 \end{bmatrix} = \begin{bmatrix} 22 & 19 \\ 22 & 2 \end{bmatrix} \quad \dots(2) \end{aligned}$$

From (1) and (2), we find that $(AB)' = B'A'$.

Example 3. If $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $B = [-2 \ -1 \ -4]$, verify that $(AB)' = B'A'$.

$$\text{Solution. } AB = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} [-2 \ -1 \ -4]$$

$$= \begin{bmatrix} (-1).(-2) & (-1).(-1) & (-1).(-4) \\ 2.(-2) & 2.(-1) & 2.(-4) \\ 3.(-2) & 3.(-1) & 3.(-4) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ -4 & -2 & -8 \\ -6 & -3 & -12 \end{bmatrix}$$

$$\Rightarrow (AB)' = \begin{bmatrix} 2 & 1 & 4 \\ -4 & -2 & -8 \\ -6 & -3 & -12 \end{bmatrix}' = \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix} \quad \dots(i)$$

$$\text{Also } B'A' = [-2 \ -1 \ -4]' \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}' = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix} [-1 \ 2 \ 3]$$

$$= \begin{bmatrix} (-2).(-1) & (-2).2 & (-2).3 \\ (-1).(-1) & (-1).2 & (-1).3 \\ (-4).(-1) & (-4).2 & (-4).3 \end{bmatrix} = \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix} \quad \dots(ii)$$

From (i) and (ii), we find that $(AB)' = B'A'$.

Example 4. Find the values of x, y, z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfies the equation $A'A = I_3$.

$$\text{Solution. Given } A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$$

$$\text{Now } A'A = I_3 \Rightarrow \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow 2x^2 = 1, 6y^2 = 1, 3z^2 = 1$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{2}}, y = \pm \frac{1}{\sqrt{6}}, z = \pm \frac{1}{\sqrt{3}}.$$

Example 5. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is matrix which satisfies $AA^t = 9I_3$, then find the values of a and b .

Solution. $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$

$$\text{Given } AA^t = 9I_3 \Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow a+4+2b=0, 2a+2-2b=0, a^2+4+b^2=9$$

$$\Rightarrow a+2b+4=0 \dots(i) \quad a-b+1=0 \dots(ii) \quad a^2+b^2=5 \dots$$

Solving (i) and (ii) simultaneously for a and b , we get

$$a = -2, b = -1$$

We note that these values of a and b satisfy (iii).

Hence, $a = -2$ and $b = -1$.

EXERCISE 3.4

1. If $A = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 4 & 5 \end{bmatrix}$, then find $(3A)'$.

2. If $A = \begin{bmatrix} 2 & -1 & 5 \\ 4 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & 1 \\ -1 & 2 & -3 \end{bmatrix}$, find $A' + B'$.

3. If $A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then find $A^T - B^T$.

4. Let $A = \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, verify that

$$(i) (A')' = A \quad (ii) (3A)' = 3A' \quad (iii) (A - B)' = A' - B' \quad (iv) (AB)' = B'A'$$

5. If $A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, then verify that

$$(i) (A')' = A \quad (ii) (A + B)' = A' + B'$$

$$(iii) (kB)' = kB', \text{ where } k \text{ is any real number.}$$

6. If $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, then find $(A + 2B)'$.

7. If $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$ and $B = [-1 \ 2 \ 1]$, verify that $(AB)' = B'A'$.

8. If $A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}$, find $(BA)'$.

9. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & x \\ -2 & 2 & -1 \end{bmatrix}$ is a matrix satisfying $AA' = 9I_3$, find x .

10. Find the integral value of x if $[x \ 4 \ -1] \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 0 \\ 2 & 2 & 4 \end{bmatrix} [x \ 4 \ -1]^t = 0$.

Answers

1. $\begin{bmatrix} 6 & -3 \\ -9 & 12 \\ 0 & 15 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 3 \\ 2 & 2 \\ 6 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$

6. $\begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$

8. $\begin{bmatrix} -3 & 1 & 5 \\ -2 & 9 & 10 \end{bmatrix}$

9. -2

10. -4

SYMMETRIC AND SKEW-SYMMETRIC MATRICES

Symmetric matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is called **symmetric** iff $A' = A$ i.e.

iff $a_{ji} = a_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq n$.

For example, $\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$ and $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ are symmetric matrices of orders 2 and 3 respectively.

Skew-symmetric matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is called **skew-symmetric** iff $A' = -A$ i.e.

iff $a_{ji} = -a_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq n$.

Remark

Every element in the principal diagonal of a skew-symmetric matrix is zero, for $A' = -A$

$$\Rightarrow a_{ji} = -a_{ij} \text{ for all } i, j \Rightarrow a_{ii} = -a_{ii} \text{ for all } i$$

$$\Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0.$$

For example : $\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ are skew-symmetric matrices of orders 2 and 3 respectively.

Every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix

Proof. Let A be a square matrix, then

$$A = \frac{1}{2}A + \frac{1}{2}A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

$$\Rightarrow A = P + Q \text{ (say)}$$

$$\text{where } P = \frac{1}{2}(A + A') \text{ and } Q = \frac{1}{2}(A - A')$$

We note that

$$P' = \left(\frac{1}{2}(A + A') \right)' = \frac{1}{2}(A + A')' = \frac{1}{2}(A' + (A')')$$

$$= \frac{1}{2}(A' + A) = \frac{1}{2}(A + A') = P \text{ and}$$

$$Q' = \left(\frac{1}{2}(A - A') \right)' = \frac{1}{2}(A - A')' = \frac{1}{2}(A' - (A')')$$

$$= \frac{1}{2}(A' - A) = -\frac{1}{2}(A - A') = -Q$$

$\Rightarrow P$ is symmetric and Q is skew-symmetric.

Therefore, A is the sum of a symmetric matrix P and a skew-symmetric matrix Q .

To prove uniqueness

If possible, let $A = R + S$ be any other such representation where R is symmetric matrix and S is skew-symmetric matrix. As R is symmetric and S is skew-symmetric

$$\Rightarrow R' = R \text{ and } S' = -S$$

$$\Rightarrow A' = (R + S)' = R' + S' = R - S$$

$$\therefore A + A' = (R + S) + (R - S) = 2R \text{ and}$$

$$A - A' = (R + S) - (R - S) = 2S$$

$$\Rightarrow R = \frac{1}{2}(A + A') \text{ and } S = \frac{1}{2}(A - A')$$

$\Rightarrow$ the representation (i) of A as the sum of a symmetric and a skew-symmetric matrix is unique.

ILLUSTRATIVE EXAMPLES

Example 1. If $A = \begin{bmatrix} 4 & 1 \\ 5 & 8 \end{bmatrix}$, show that $A + A^T$ is a symmetric matrix, where A^T denotes the transpose matrix A .

$$\text{Solution. Let } P = A + A^T = \begin{bmatrix} 4 & 1 \\ 5 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 5 & 8 \end{bmatrix}^T = \begin{bmatrix} 4 & 1 \\ 5 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 1 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 4+4 & 1+5 \\ 5+1 & 8+8 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 6 & 16 \end{bmatrix}$$

$$\Rightarrow P^T = \begin{bmatrix} 8 & 6 \\ 6 & 16 \end{bmatrix}^T = \begin{bmatrix} 8 & 6 \\ 6 & 16 \end{bmatrix} = P$$

(from

$\Rightarrow P$ is symmetric $\Rightarrow A + A^T$ is a symmetric matrix.

Example 2. If $A = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix}$, show that $A - A^T$ is skew-symmetric matrix, where A^T is the transpose of the matrix A .

Solution. Let $P = A - A^T = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 1-1 & 4-3 \\ 3-4 & 7-7 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \dots(i)$$

$$\Rightarrow P^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\Rightarrow P^T = -P$$

$\Rightarrow P$ is skew-symmetric

$\Rightarrow A - A^T$ is skew-symmetric matrix.

(from (i))

Example 3. Show that the matrix, $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, is a skew-symmetric matrix.

Solution. Given $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

$$\Rightarrow A' = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}' = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = -A$$

$\Rightarrow A$ is a skew-symmetric matrix.

Example 4. If the matrix $\begin{bmatrix} -2 & x-y & 5 \\ 1 & 0 & 4 \\ x+y & z & 7 \end{bmatrix}$ is symmetric, find the values of x , y and z .

Solution. We know that a square matrix A is symmetric iff $A' = A$.

$$\text{Given } \begin{bmatrix} -2 & x-y & 5 \\ 1 & 0 & 4 \\ x+y & z & 7 \end{bmatrix} \text{ is symmetric } \Rightarrow \begin{bmatrix} -2 & 1 & x+y \\ x-y & 0 & z \\ 5 & 4 & 7 \end{bmatrix} = \begin{bmatrix} -2 & x-y & 5 \\ 1 & 0 & 4 \\ x+y & z & 7 \end{bmatrix}$$

$$\Rightarrow x-y=1, x+y=5, z=4 \Rightarrow x=3, y=2, z=4.$$

Example 5. (i) For what value of x , is the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$ a skew-symmetric matrix?

(ii) If the matrix $\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ is skew-symmetric, find the values of a , b and c .

Solution. (i) We know that a square matrix A is skew-symmetric iff $A' = -A$.

$$\text{Given } \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix} \text{ is skew-symmetric } \Rightarrow \begin{bmatrix} 0 & -1 & x \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & x \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -x & 3 & 0 \end{bmatrix} \Rightarrow x = 2, -x = -2 \Rightarrow x = 2.$$

(ii) We know that a square matrix A is skew-symmetric iff $A' = -A$.

$$\text{Given } \begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix} \text{ is skew-symmetric} \Rightarrow \begin{bmatrix} 0 & 2 & c \\ a & b & 1 \\ 3 & -1 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 2 & c \\ a & b & 1 \\ 3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a & -3 \\ -2 & -b & 1 \\ -c & -1 & 0 \end{bmatrix} \Rightarrow 2 = -a, c = -3, b = -b$$

$$\Rightarrow a = -2, c = -3, 2b = 0 \Rightarrow a = -2, b = 0, c = -3.$$

Example 6. If A is any square matrix, prove that AA' is symmetric.

$$\begin{aligned} \text{Solution. } (AA')' &= (A')'A' \\ &= AA' \end{aligned}$$

$\Rightarrow AA'$ is a symmetric matrix.

Example 7. If A, B are square matrices of equal order and B is symmetric, then show that $A'BA$ is also symmetric.

Solution. A, B are square matrices of same order $\Rightarrow A'BA$ is defined.

$$B \text{ is symmetric} \Rightarrow B' = B.$$

$$\text{Now } (A'BA)' = A'B'(A')' = A'BA \Rightarrow A'BA \text{ is symmetric.}$$

Example 8. If A is a square matrix and P is any square matrix of order equal to that of A , prove that $P'AP$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.

Solution. As A and P are square matrices of equal order, therefore, $P'AP$ is defined.

Let A be symmetric, then $A' = A$.

$$\text{Now } (P'AP)' = P'A'(P')' = P'AP \Rightarrow P'AP \text{ is a symmetric matrix.}$$

Let A be skew-symmetric, then $A' = -A$.

$$\text{Now } (P'AP)' = P'A'(P')' = P'(-A)P = -(P'AP) \Rightarrow P'AP \text{ is skew-symmetric.}$$

Hence, matrix $P'AP$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.

Example 9. If a matrix is both symmetric and skew-symmetric, then show that it is a null matrix.

Solution. Let matrix A be both symmetric and skew-symmetric.

As A is symmetric, $A' = A$

Also A is skew-symmetric $\Rightarrow A' = -A$

From (i) and (ii), we get

$$A = -A \Rightarrow 2A = O \Rightarrow A = O.$$

Hence, matrix A is a null matrix.

Example 10. If A and B are symmetric matrices of the same order, prove that AB is symmetric if and only if A and B commute.

Solution. Since A, B are symmetric, $A' = A$ and $B' = B$.

As A, B are square matrices of same order, AB and BA are both defined.

Now AB is symmetric iff $(AB)' = AB$ i.e. iff $B'A' = AB$

i.e. iff $BA = AB$ i.e. iff A and B commute.

Example 11. If A, B are skew-symmetric matrices of the same order, prove that AB is symmetric iff A and B commute.

Solution. As A and B are skew-symmetric matrices, therefore, $A' = -A$ and $B' = -B$... (i)

Since A and B are square matrices of same order, AB and BA are both defined.

Now, AB is symmetric iff $(AB)' = AB$

$$\text{i.e. iff } B'A' = AB \quad (\text{reversal law for transposes})$$

$$\text{i.e. iff } (-B)(-A) = AB \quad (\text{using (i)})$$

$$\text{i.e. iff } BA = AB \quad \text{i.e. iff } A \text{ and } B \text{ commute.}$$

Example 12. If A, B are symmetric matrices of same order, prove that $AB - BA$ is a skew-symmetric matrix.

Solution. As A and B are symmetric matrices, therefore, $A' = A$ and $B' = B$.

Since A and B are square matrices of same order, AB, BA and $AB - BA$ are all defined.

$$\text{Now, } (AB - BA)' = (AB)' - (BA)' \quad (\because (A - B)' = A' - B')$$

$$= B'A' - A'B' \quad (\text{reversal law})$$

$$= BA - AB \quad (\because A' = A \text{ and } B' = B)$$

$$= -(AB - BA)$$

$$\Rightarrow AB - BA \text{ is a skew-symmetric matrix} \quad (\because A' = -A)$$

Example 13. If A and B are square matrices of same order, then prove that $AB' - BA'$ is a skew-symmetric matrix.

Solution. Since A, B are square matrices of same order, AB', BA' and $AB' - BA'$ are all defined.

$$\text{Now } (AB' - BA')' = (AB')' - (BA')' \quad (\because (A - B)' = A' - B')$$

$$= (B')'A' - (A')'B' \quad (\text{reversal law})$$

$$= BA' - AB' \quad (\because (A')' = A)$$

$$= -(AB' - BA')$$

$$\Rightarrow AB' - BA' \text{ is a skew-symmetric matrix} \quad (\because A' = -A)$$

Example 14. Express the following matrix as the sum of a symmetric matrix and a skew-symmetric matrix and verify your result :

$$\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

$$\text{Solution. Let } A = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

$$\therefore A + A' = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2} (A + A') = \frac{1}{2} \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix},$$

which is clearly a symmetric matrix and

$$A - A' = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2} (A - A') = \frac{1}{2} \begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{5}{2} & -\frac{3}{2} \\ \frac{5}{2} & 0 & -3 \\ \frac{3}{2} & 3 & 0 \end{bmatrix}$$

which is clearly a skew-symmetric matrix.

$$\text{Since } A = \frac{1}{2} (A + A') + \frac{1}{2} (A - A')$$

$$\therefore A = \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{5}{2} & -\frac{3}{2} \\ \frac{5}{2} & 0 & -3 \\ \frac{3}{2} & 3 & 0 \end{bmatrix}$$

Thus A has been expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Verification

$$\begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{5}{2} & -\frac{3}{2} \\ \frac{5}{2} & 0 & -3 \\ \frac{3}{2} & 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} = A.$$

EXERCISE 3.5

1. Show that $(A + A')$ is a symmetric matrix, if $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$.
2. If $A = \begin{bmatrix} 5 & -1 \\ -2 & 6 \end{bmatrix}$, determine whether $A - A^T$ is symmetric or skew-symmetric.
3. (i) If $A = \begin{bmatrix} 5 & a \\ b & 0 \end{bmatrix}$ and A is symmetric matrix, show that $a = b$.
(ii) If the matrix $\begin{bmatrix} 6 & -x^2 \\ 2x - 15 & 10 \end{bmatrix}$ is symmetric, find the value(s) of x.
4. If A is a square matrix, prove that $A'A$ is symmetric.
5. If A, B are symmetric matrices and $AB = BA$, then show that AB is symmetric.
6. If A, B and AB are all symmetric matrices, then show that $AB = BA$.
7. If A, B are skew-symmetric matrices and $AB = BA$, then show that AB is symmetric.
8. If A, B are skew-symmetric matrices of same order and AB is symmetric, then show that $AB = BA$.

9. Show that the matrix A, where $A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$, is a symmetric matrix.

Hint. Show that $A' = A$.

10. If the matrix $\begin{bmatrix} 0 & 2b & -2 \\ 3 & 1 & 3 \\ 3a & 3 & -1 \end{bmatrix}$ is symmetric, find the values of a and b .

11. Find the value of $2a + 3b - c$, if $A = \begin{bmatrix} 0 & -1 & 28 \\ a-8 & 0 & 3b \\ -c+2 & -2 & 0 \end{bmatrix}$ is a skew-symmetric matrix.

12. If A and B are symmetric matrices of the same order, prove that $AB + BA$ is symmetric.

Hint. Prove that $(AB + BA)' = AB + BA$.

13. Express each of the following matrices as the sum of a symmetric and a skew-symmetric matrices :

(i) $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$

(ii) $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

14. Express the following matrices as the sum of symmetric and skew-symmetric matrices:

(i) $\begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 4 & -6 \\ 7 & 3 & 5 \\ 1 & -2 & 4 \end{bmatrix}$

Answers

2. Skew-symmetric

3. (ii) 3, -5

10. $a = -\frac{2}{3}$; $b = \frac{3}{2}$

11. -10

13. (i) $\begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 3 & -\frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}$

14. (i) $\begin{bmatrix} 3 & 3 & \frac{5}{2} \\ 3 & 1 & \frac{9}{2} \\ \frac{5}{2} & \frac{9}{2} & 7 \end{bmatrix} + \begin{bmatrix} 0 & -1 & \frac{5}{2} \\ 1 & 0 & -\frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & \frac{11}{2} & -\frac{5}{2} \\ \frac{11}{2} & 3 & \frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 4 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{3}{2} & -\frac{7}{2} \\ \frac{3}{2} & 0 & \frac{7}{2} \\ \frac{7}{2} & -\frac{7}{2} & 0 \end{bmatrix}$

MULTIPLE CHOICE QUESTIONS

Choose the correct answer from the given four options in questions (1 to 23):

1. The matrix $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ is a

(a) scalar matrix (b) diagonal matrix (c) square matrix (d) none of these

2. If $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 5 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & -6 \end{bmatrix}$, then

(a) only AB is defined

(b) only BA is defined

(c) AB and BA are both defined

(d) AB and BA both are not defined

3. If A is any $m \times n$ matrix and B is a matrix such that AB and BA are both defined, then B is a matrix of order

(a) $n \times n$

(b) $m \times m$

(c) $m \times n$

(d) $n \times m$

4. If $A = \begin{bmatrix} -3 & x \\ y & 5 \end{bmatrix}$ and $A = A'$, then

(a) $x = 5, y = -3$

(b) $x = -3, y = 5$

(c) $x = y$

(d) none of these

5. If $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $A^2 - 6A$ is equal to

(a) $3I$

(b) $-5I$

(c) $5I$

(d) none of these

6. If $A = [a_{ij}]_{2 \times 2}$ where $a_{ij} = \begin{cases} 1, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases}$, then A^2 is equal to

(a) I

(b) A

(c) O

(d) none of these

7. The number of all possible matrices of order 3×3 with each entry 0 or 1 is

(a) 18

(b) 27

(c) 81

(d) 512

8. If A and B are square matrices of same order, then $(A + B)(A - B)$ is equal to

(a) $A^2 - B^2$

(b) $A^2 - BA - AB + B^2$

(c) $A^2 - AB + BA - B^2$

(d) $A^2 + AB - BA - B^2$

9. If A is a square matrix such that $A^2 = I$, then $(A + I)^3 + (A - I)^3 - 7A$ is equal to

(a) A

(b) $A - I$

(c) $A + I$

(d) $3A$

10. If $A = \begin{bmatrix} \frac{1}{3} & 2 \\ 0 & 2x - 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$ and $AB = I_2$, then the value of x is

(a) -1

(b) 0

(c) 1

(d) 2

11. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $A^2 - xA = I_2$, then the value of x is

(a) 4

(b) 2

(c) -2

(d) -4

12. If $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$, then

(a) $\alpha = a^2 + b^2, \beta = ab$

(b) $\alpha = a^2 + b^2, \beta = 2ab$

(c) $\alpha = a^2 + b^2, \beta = a^2 - b^2$

(d) $\alpha = 2ab, \beta = a^2 + b^2$

13. If matrix $A = [a_{ij}]_{2 \times 2}$, where $a_{ij} = \begin{cases} 1, & i \neq j \\ 0, & i = j \end{cases}$, then A^3 is equal to
 (a) A (b) I (c) O (d) none of these
14. If A is a square matrix such that $A^2 = A$, then $(I - A)^3 + A$ is equal to
 (a) I (b) O (c) $I - A$ (d) $I + A$
15. If A and B are square matrices of same order, then $AB' - BA'$ is a
 (a) skew-symmetric matrix (b) symmetric matrix
 (c) null matrix (d) unit matrix
16. If A and B are symmetric matrices of same order, then $AB - BA$ is a
 (a) symmetric matrix (b) skew-symmetric matrix
 (c) zero matrix (d) identity matrix
17. The matrix $\begin{bmatrix} 0 & -5 & 3 \\ 5 & 0 & -7 \\ -3 & 7 & 0 \end{bmatrix}$ is a
 (a) diagonal matrix (b) symmetric matrix
 (c) skew-symmetric matrix (d) scalar matrix
18. If the matrix $\begin{bmatrix} 0 & -1 & 3x \\ 1 & y & -5 \\ -6 & 5 & 0 \end{bmatrix}$ is skew-symmetric, then
 (a) $x = -2, y = 0$ (b) $x = 2, y = 0$ (c) $x = -2, y = 1$ (d) $x = 2, y = -1$
19. If $A = \begin{bmatrix} 0 & a & 1 \\ -1 & b & 1 \\ -1 & c & 0 \end{bmatrix}$ is a skew-symmetric matrix, then the value of $(a + b + c)^2$ is
 (a) 1 (b) 0 (c) 4 (d) none of these
20. If matrix A is given by $A = [a_{ij}]_{2 \times 2}$, where $a_{ij} = i + j$, then A is equal to
 (a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$
 (CBSE 2022 Term I)
21. If A is a square matrix such that $A^2 = A$, then $(I + A)^2 - 3A$ is equal to
 (a) I (b) $2A$ (c) $3I$ (d) A
 (CBSE 2022 Term I)
22. If $A = \begin{bmatrix} 1 & 2 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $AB = I_3$, then $(x + y)$ equals
 (a) 0 (b) -1 (c) 2 (d) -2
 (CBSE 2022 Term I)
23. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ a & b & 2 \end{bmatrix}$, then $aI + bA + 2A^2$ equals
 (a) A (b) $-A$ (c) abA (d) none of these
 (CBSE 2022 Term I)

Answers

- | | | | | | |
|---------|---------|---------|---------|---------|---------|
| 1. (c) | 2. (c) | 3. (d) | 4. (c) | 5. (b) | 6. (a) |
| 7. (d) | 8. (c) | 9. (a) | 10. (c) | 11. (a) | 12. (b) |
| 13. (a) | 14. (a) | 15. (a) | 16. (b) | 17. (c) | 18. (b) |
| 19. (b) | 20. (b) | 21. (a) | 22. (a) | 23. (d) | |

Assertion-Reason Type Questions (Solved)

These type of questions consist of two Statements.

Statement I is called Assertion (A) and Statement II is called Reason (R). Read the given Statements carefully and choose the correct answer from the options given below:

- Both the Statements are true and Statement II is the correct explanation of Statement I.
- Both the Statements are true and Statement II is not the correct explanation of Statement I.
- Statement I is true, Statement II is false.
- Statement I is false, Statement II is true.

- Let A and B be symmetric matrices of the same order.

Statement I: $AB - BA$ is skew-symmetric

Statement II: $AB + BA$ is symmetric.

- Let A and B be two symmetric matrices of order 3.

Statement I: $A(BA)$ and $(AB)A$ are symmetric matrices

Statement II: AB is symmetric matrix if $AB = BA$.

- Let A and B be two square matrices of the same order such that $AB = BA$.

Statement I: $(A + B)^2 = A^2 + 2AB + B^2$

Statement II: $(A + B)(A - B) = A^2 - B^2$

- Let A , B and C are three square matrices of the same order.

Statement I: If $AB = O$, then $A = O$ or $B = O$

Statement II: $A + B = A + C \Rightarrow B = C$.

Solutions to Assertion-Reason Type Questions

- Given A and B are symmetric matrices of the same order

$$\Rightarrow A' = A \text{ and } B' = B.$$

$$\text{Now, } (AB - BA)' = (AB)' - (BA)' = B'A' - A'B'$$

$$= BA - AB = -(AB - BA)$$

$$\Rightarrow (AB - BA) \text{ is skew-symmetric.}$$

$$\text{Also, } (AB + BA)' = (AB)' + (BA)' = B'A' + A'B'$$

$$= BA + AB = (AB + BA)$$

$$\Rightarrow (AB + BA) \text{ is symmetric.}$$

$\therefore$ Both Statements I and II are true but Statement II is not the correct explanation of Statement I.

$\therefore$ Option (b) is the correct answer.

- Given A and B are symmetric matrices of order 3

$$\Rightarrow A' = A \text{ and } B' = B.$$

$$\text{Now, } (A(BA))' = (BA)' \cdot A' = (A'B') \cdot A'$$

$$= (AB)A = A(BA)$$

(using Associative law)

$$= A(BA) \text{ is symmetric}$$

$$\text{and } ((AB)A)' = A'(AB)' = A'(B'A')$$

$$= A(BA) = (AB)A$$

(using Associative law)

$$\Rightarrow (AB)A \text{ is symmetric.}$$

$\therefore$ Statement I is true.

$$\text{Also, } (AB)' = B'A' = BA$$

$$\therefore (AB) \text{ is symmetric if } (AB)' = AB$$

$$\Rightarrow AB = BA$$

$\therefore$ Statement II is true.

Hence, both Statements I and II are true but Statement II is not the correct explanation of Statement I.

$\therefore$ Option (b) is the correct answer.

3. Given A and B are two square matrices of same order such that $AB = BA$

$$(A + B)^2 = (A + B)(A + B) = AA + AB + BA + BB$$

(using Distributive law)

$$= A^2 + AB + AB + B^2$$

($\because AB = BA$)

$$= A^2 + 2AB + B^2$$

$\therefore$ Statement I is true.

$$\text{Also, } (A + B)(A - B) = AA - AB + BA - BB$$

(using Distributive law)

$$= A^2 - AB + AB - B^2$$

($\because AB = BA$)

$$= A^2 - B^2$$

$\therefore$ Statement II is true.

Hence, both the Statements I and II are true but Statement II is not the correct explanation of Statement I.

$\therefore$ Option (b) is the correct answer.

4. Statement I is not true because

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq O \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \neq O.$$

$$\text{But } A \cdot B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

$$\text{Also, } A + B = A + C \Rightarrow -A + (A + B) = -A + (A + C)$$

$$\Rightarrow (-A + A) + B = (-A + A) + C \Rightarrow O + B = O + C$$

$$\Rightarrow B = C$$

$\therefore$ Statement II is true.

$\therefore$ Option (d) is the correct answer.

Assertion-Reason Type Questions (Unsolved)

These type of questions consist of two Statements.

Statement I is called Assertion (A) and Statement II is called Reason (R). Read the given Statements carefully and choose the correct answer from the options given below:

(a) Both the Statements are true and Statement II is the correct explanation of Statement I.

- (b) Both the Statements are true and Statement II is not the correct explanation of Statement I.
 (c) Statement I is true, Statement II is false.
 (d) Statement I is false, Statement II is true.

1. Let A be a square matrix.

Statement I: Every square matrix can be expressed as the sum of symmetric and skew-symmetric matrices.

Statement II: $(A + A')$ is symmetric matrix and $(A - A')$ is skew-symmetric matrix.

2. Let A and B be two square matrices of the same order.

Statement I: $(A'BA)$ is symmetric if B is symmetric.

Statement II: $(A'BA)$ is skew-symmetric if B is skew-symmetric.

3. **Statement I:** If $A = \begin{bmatrix} 0 & a & -2 \\ -3 & 0 & 4 \\ b & c & 0 \end{bmatrix}$ is a skew-symmetric matrix, then $a + b + c = 1$.

Statement II: For a skew-symmetric matrix A, $A' = -A$.

4. **Statement I:** If A, B and C are the matrices of order $3 \times l$, $2 \times m$ and $3 \times n$ respectively, then $(CA + BA)$ is defined, if $n = 3$ and $m = 3$.

Statement II: Multiplication of two matrices A and B i.e. AB is defined if number of columns of A = number of rows of B.

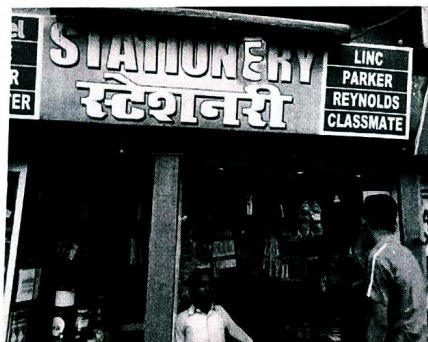
Answers

1. (a) 2. (b) 3. (a) 4. (d)

CASE-STUDY BASED QUESTIONS (SOLVED)

Case-Study 1

A manufacture produces three stationery products pencil, eraser and sharpener which he sells in two markets. Annual sales are indicated below:



Market	Products (in numbers)		
	Pencil	Eraser	Sharpener
A	10000	2000	18000
B	6000	20000	8000

If the unit sale price of pencil, eraser and sharpener are ₹2.50, ₹1.50 and ₹1.00 respectively, and unit cost of the above three commodities are ₹2.00, ₹1.00 and ₹0.50 respectively.

Based on the above information, answer the following questions:

- (a) Find the total revenue of market A

(b) Find the total revenue of market B

(c) Find the profit in market A

OR

Find the profit in market

Solution. Let A represent the annual sales of different products in market A, then

$$\begin{array}{ccc} \text{Pencil} & \text{Eraser} & \text{Sharpener} \\ A = (10000 & 2000 & 18000) \end{array}$$

Let B represent the annual sales of different products in market B, then

$$\begin{array}{ccc} \text{Pencil} & \text{Eraser} & \text{Sharpener} \\ B = (6000 & 20000 & 8000) \end{array}$$

R represents the selling price of different products, then

$$R = \begin{pmatrix} ₹ 2.50 \\ ₹ 1.50 \\ ₹ 1.00 \end{pmatrix} \begin{array}{l} \rightarrow \text{Pencil} \\ \rightarrow \text{Eraser} \\ \rightarrow \text{Sharpener} \end{array}$$

C represents the cost price of different products, then

$$C = \begin{pmatrix} ₹ 2.00 \\ ₹ 1.00 \\ ₹ 0.50 \end{pmatrix} \begin{array}{l} \rightarrow \text{Pencil} \\ \rightarrow \text{Eraser} \\ \rightarrow \text{Sharpener} \end{array}$$

$$\begin{aligned} (a) \text{ Total revenue of market A} = AR &= (10000 \ 2000 \ 18000) \begin{pmatrix} ₹ 2.50 \\ ₹ 1.50 \\ ₹ 1.00 \end{pmatrix} \\ &= (₹25000 + ₹3000 + ₹18000) = (₹46000) \end{aligned}$$

$$\begin{aligned} (b) \text{ Total revenue of market B} = BR &= (6000 \ 20000 \ 8000) \begin{pmatrix} ₹ 2.50 \\ ₹ 1.50 \\ ₹ 1.00 \end{pmatrix} \\ &= (₹15000 + ₹30000 + ₹8000) = (₹53000) \end{aligned}$$

$$\begin{aligned} (c) \text{ Cost incurred in market A} = AC &= (10000 \ 2000 \ 18000) \begin{pmatrix} ₹ 2.00 \\ ₹ 1.00 \\ ₹ 0.50 \end{pmatrix} \\ &= (₹20000 + ₹2000 + ₹9000) = (₹31000) \end{aligned}$$

OR

Profit in market A = AR - AC

$$= (₹46000 - ₹31000) = (₹15000)$$

$$\text{Cost incurred in market B} = BC = (6000 \ 20000 \ 8000) \begin{pmatrix} ₹ 2.00 \\ ₹ 1.00 \\ ₹ 0.50 \end{pmatrix}$$

$$= (₹12000 + ₹20000 + ₹4000) = (₹36000)$$

So, profit in market B = BR - BC

$$= (₹53000 - ₹36000) = (₹17000)$$

Case-Study 2

Amit, Biraj and Chirag were given the task of creating a square matrix of order 2.

Below are the matrices created by them. A, B, C are the matrices created by Amit, Biraj and Chirag respectively

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}.$$

If $a = 4$ and $b = -2$, based on the above information, answer the following questions:

- Find the sum of the matrices A, B and C.
- Find $(A^T)^T$.
- Find $AC - BC$.

OR

Find $(a + b)B$.

Solution. Given $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$

$$\begin{aligned} (a) \quad A + (B + C) &= A + \left(\begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 1 & 6 \end{bmatrix} \end{aligned}$$

$$(b) \quad (A^T)^T = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = A.$$

$$(c) \quad AC = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 1 & -6 \end{bmatrix}$$

$$BC = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 7 & -10 \end{bmatrix}$$

$$\therefore AC - BC = \begin{bmatrix} 4 & -4 \\ 1 & -6 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 7 & -10 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -6 & 4 \end{bmatrix}$$

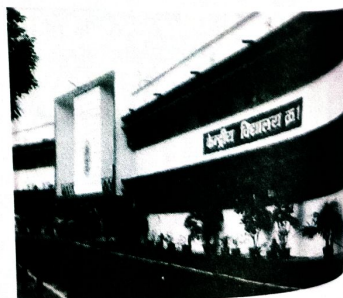
OR

$$(a + b)B = (4 + (-2))B = 2B = 2 \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 2 & 10 \end{bmatrix}$$

Case-Study 3

Three schools DPS, CVC and KVS decided to organize a fair for collecting money for helping the flood victims. They sold handmade fans, mats and plates from recycled material at a cost of ₹25, ₹100 and ₹

each respectively. The numbers of articles sold are given as



School/Article	DPS	CVC	KVS
Handmade fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Based on the information given above, answer the following questions:

- What is the total money (in ₹) collected by the school DPS?
- What is the total amount of money (in ₹) collected by schools CVC and KVS?
- What is the total amount of money collected by all three schools DPS, CVC and KVS?

OR

If the number of handmade fans and plates are interchanged for all the schools, then what is the total money collected by all schools?

Solution Let matrices D, C and K represent the number of articles sold by three schools DPS, CVC and KVS respectively, then

$$D = \begin{pmatrix} 40 \\ 50 \\ 20 \end{pmatrix}, C = \begin{pmatrix} 25 \\ 40 \\ 30 \end{pmatrix} \text{ and } K = \begin{pmatrix} 35 \\ 50 \\ 40 \end{pmatrix}.$$

Let matrix M represent the selling price of different items, then

$$M = (\text{₹ } 25 \quad \text{₹ } 100 \quad \text{₹ } 50)$$

- Total money collected by DPS = MD

$$= (\text{₹ } 25 \quad \text{₹ } 100 \quad \text{₹ } 50) \begin{pmatrix} 40 \\ 50 \\ 20 \end{pmatrix}$$

$$= (\text{₹ } 1000 + \text{₹ } 5000 + \text{₹ } 1000) = (\text{₹ } 7000)$$

- Total money collected by schools CVC and KVS

$$= M (C + K) = (\text{₹ } 25 \quad \text{₹ } 100 \quad \text{₹ } 50) \left[\begin{pmatrix} 25 \\ 40 \\ 30 \end{pmatrix} + \begin{pmatrix} 35 \\ 50 \\ 40 \end{pmatrix} \right]$$

$$= (\text{₹ } 25 \quad \text{₹ } 100 \quad \text{₹ } 50) \begin{pmatrix} 60 \\ 90 \\ 70 \end{pmatrix}$$

$$= (\text{₹ } 1500 + \text{₹ } 9000 + \text{₹ } 3500) = (\text{₹ } 14000)$$

- Total amount of money collected by all three schools DPS, CVC and KVS

$$= (\text{₹ } 7000) + (\text{₹ } 14000) = (\text{₹ } 21000)$$

OR

If number of handmade fans and plates are interchanged for all schools, then

$$\text{New } D = \begin{pmatrix} 20 \\ 50 \\ 40 \end{pmatrix}, \text{New } C = \begin{pmatrix} 30 \\ 40 \\ 25 \end{pmatrix}, \text{New } K = \begin{pmatrix} 40 \\ 50 \\ 35 \end{pmatrix}.$$

∴ Total money collected by all three schools

$$= M (\text{New D} + \text{New C} + \text{New K})$$

$$= (\text{₹}25 \quad \text{₹}100 \quad \text{₹}50) \left[\begin{pmatrix} 20 \\ 50 \\ 40 \end{pmatrix} + \begin{pmatrix} 30 \\ 40 \\ 25 \end{pmatrix} + \begin{pmatrix} 40 \\ 50 \\ 35 \end{pmatrix} \right]$$

$$= (\text{₹}25 \quad \text{₹}100 \quad \text{₹}50) \begin{pmatrix} 90 \\ 140 \\ 100 \end{pmatrix}$$

$$= (\text{₹}2250 + \text{₹}14000 + \text{₹}5000) = (\text{₹}21250)$$

CASE-STUDY BASED QUESTION (UNSOLVED)

Three students Ram, Mohan and Ankit go to a shop to buy stationary. Ram purchases 2 dozen notebooks, 1 dozen pens and 4 pencils, Mohan purchases 1 dozen notebook, and 8 pencils and Ankit purchases 6 notebooks, 4 pens and 6 pencils. A notebook costs ₹4.50 and a pencil costs ₹1.50.

Let A and B be the matrices representing the number of items purchased by the three students and the prices of the items respectively.



Based on the above information, answer the following questions:

- Find the order of matrix B representing the price of three items
- Find the order of matrix A representing items purchased by three students
- Find the bill amount of Ram

OR

Find the total bill amount of all three students.

Answer

- (a) 3×1 (b) 3×3 (c) ₹420 OR ₹756